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A BANACH LATTICE WITHOUT THE APPROXIMATION PROPERTY

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ABSTRACT
A Banach lattice L without the approximation property is constructed. The construction can be improved so that L is, in addition, uniformly convex. These results yield the existence of a uniformly convex Banach space with symmetric basis and without the uniform approximation property.

A Banach space L is said to have the *approximation property* (AP) if the identity operator on L can be approximated uniformly on every compact subset of L by bounded finite rank operators, i.e. if for every compact $K \subset L$ and for every $\varepsilon > 0$ there exists $T: L \to L$ with $\operatorname{rk} T = \dim TL < \infty$ and such that $\| Tx - x \| < \varepsilon$ for every $x \in K$.

A *Banach lattice* L is a partially ordered real Banach space for which

(i) $x \leqq y$ implies $x + z \leqq y + z$ for every $x, y, z \in L$,

(ii) $\alpha x \geqq 0$ whenever $x \geqq 0$ and $\alpha \geqq 0$,

(iii) the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ exist for every $x, y \in L$,

(iv) $\| x \| \leqq \| y \|$ whenever $| x | \leqq | y |$ (where $| x | = x \vee (-x)$).

1. Construction of a Banach lattice without AP

Let $I = [0, 1]$, let λ be the Lebesgue measure on I. Let J_n denote the set of intervals $\{[k \cdot 2^{-n}, (k+1)2^{-n}]: k = 0, 1, \cdots, 2^n - 1\}$. By B_n we denote the (finite) σ-algebra of subsets of I generated by J_n. Let $\varphi_n: I \to I$ be a measure preserving transformation such that

$$\varphi_n([k 2^{-n}, (k+1)2^{-n}]) = \begin{cases} [(k+1)2^{-n}, (k+2)2^{-n}] & \text{if } k \text{ is even,} \\ [(k-1)2^{-n}, k 2^{-n}] & \text{if } k \text{ is odd.} \end{cases}$$

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At the very end of this chapter we shall construct a sequence Δ_n of (disjoint) partitions of I satisfying the following conditions:
(*) the elements of Δ_n are B_n-measurable subsets of I of equal length, M_n = the number of elements of Δ_n, goes faster to ∞ than any power of n.
(**) If $A \in \Delta_n$, $B \in \Delta_m$ then

$$\lambda(\varphi_n(A) \cap B) \leqq 4\lambda(A) \cdot \lambda(B).$$

Let f be a measurable function on I. We put

$$\|f\| = \sup_m \max_{B \in \Delta_m} M_m \int_B |f| \, d\lambda$$

and define $L = \{f: \|f\| < \infty\}$, equipped with the norm $\| \ \|$ (we identify functions equal a.e.)

By standard arguments one can prove that L is a Banach space, it is clearly a lattice with the natural order relation; (i)–(iv) are trivially fulfilled. We have also

$$\|f\|_1 \leqq \|f\| \leqq \|f\|_\infty \quad \text{for all } f. \tag{1}$$

Let $T \in B(L, L)$ (= bounded operators from L into L). We define (here 1_A is the indicator function of A and $(f, g) = \int fg d\lambda$):

$$\beta_n(T) = \sum_{u \in \Delta_n} (1_u, T1_u).$$

We have the following two standard lemmas.

Lemma 1. *If $T \in B(L, L)$ is compact, then* $\lim_{n \to \infty} \beta_n(T) = 0$.

Proof. First let us notice that, by the invariance of trace, if w_j, $j = 1, \cdots, 2^n$ is any system of B_n-measurable functions satisfying $(w_i, w_j) = 2^{-n}\delta_{ij}$, then also

$$\beta_n(T) = \sum_{j=1}^{2^n} (w_j, Tw_j).$$

In particular, if $f_1, \cdots, f_{2^n}$ are B_n-measurable functions such that $|f_j| \equiv 1$ and $(f_j, f_i) = \delta_{ij}$, then

$$\beta_n(T) = 2^{-n} \sum_{j=1}^{2^n} (f_j, Tf_j)$$

(take, for example, the first 2^n Walsh functions).

Notice that, by (1), $\|f_j\| = 1$ and $\|f_j\|^* = 1$ where $\|f_j\|^* = \sup\{(f_j, f): \|f\| = 1\}$.

Take an $\varepsilon > 0$. Since T is compact, we can find $x_1, \cdots, x_k \in L$ so that for every f_j there exists an i so that $\|Tf_j - x_i\| < \varepsilon/2$. Therefore

$$|\beta_n(T)| \leqq k \max_{1 \leqq i \leqq k} 2^{-n} \sum_{j=1}^{2^n} |(f_j, x_i)| + \varepsilon/2. \tag{2}$$

Since f_j are orthonormal and $x_i \in L_1$, $\lim_{j\to\infty}(f_j, x_i) = 0$ for $i = 1, \cdots, k$ and therefore, if n is big enough,

$$2^{-n} \sum_{j=1}^{2^n} |(f_j, x_i)| < \varepsilon/2k \quad \text{for } i = 1, \cdots, k.$$

Consequently, $|\beta_n(T)| < \varepsilon$ for sufficiently big n. This proves Lemma 1.

LEMMA 2. *Assume that for every n there exists a finite set F_n so that (here $\beta_{-1} = 0$)*

$$|\beta_n(T) - \beta_{n-1}(T)| \leqq \max\{\|Tf\| : f \in F_n\}, \quad n = 0, 1, 2, \cdots \tag{3}$$

$$\sum \alpha_n < \infty \quad \textit{where} \quad \alpha_n = \max\{\|f\| : f \in F_n\}. \tag{4}$$

Then L does not have AP.

PROOF. First let us notice that, by (4), $\beta(T) = \lim_{n\to\infty} \beta_n(T)$ is well defined for all $T \in B(L, L)$.

We see immediately that β is linear and that $\beta(Id_L) = 1$, where Id_L is the identity on L.

Let $\xi_n \to \infty$ be such that we still have $\Sigma \alpha_n \xi_n < \infty$. Let us take $K = \bigcup_{n=0}^{\infty} (\xi_n \alpha_n)^{-1} F_n \cup \{0\}$. This is clearly a compact set in L, since $\cup (\xi_n \alpha_n)^{-1} F_n$ is just a sequence convergent to 0.

We have, by (4),

$$|\beta(T)| \leqq \left(\sum_{n=0}^{\infty} \alpha_n \xi_n \right) \sup\{\|Tf\| : f \in K\}.$$

Assume now that $T \in B(L, L)$ is compact and that $\|Tx - x\| < \varepsilon$ for all $x \in K$. By Lemma 1 and by linearity of β we have

$$1 = |\beta(I - T)| \leqq \left(\sum_{n=0}^{\infty} \alpha_n \xi_n \right) \cdot \varepsilon$$

and this shows that L does not even have the *compact* AP (i.e. the weaker approximation property, where the words "finite rank" in the definition of AP are replaced by "compact").

In the sequel we shall need the following

LEMMA 3. *Let $A = (a_{ij})_{i,j=1}^{m}$. There exist $\varepsilon_1, \cdots, \varepsilon_m = \pm 1$ such that*

$$\sum_{i=1}^{m} \left| \sum_{j=1}^{m} \varepsilon_j a_{ij} \right| \geqq \left| \sum_{i=1}^{m} a_{ii} \right|.$$

(Let us notice that the lemma expresses the following statement: let an operator $A: l_\infty^m \to l_1^m$ be given by $A(x) = Ax$. Then $\|A\| = \max\{\|Ax\|_1 : x \in \operatorname{ext}(\text{unit ball of } l_\infty^m)\} \geqq |\operatorname{tr} A|$.)

PROOF. We shall prove a little more than stated: there exist $\varepsilon_1, \cdots, \varepsilon_m = \pm 1$ so that $\Sigma_i \varepsilon_i \Sigma_j \varepsilon_j a_{ij} \geqq \Sigma a_{ii}$. If we put $b_{ij} = a_{ij} + a_{ji}$ this becomes

$$\sum_{1 \leqq j < i \leqq m} \varepsilon_i \varepsilon_j b_{ij} \geqq 0.$$

The last statement follows easily by induction on m:

$$\sum_{1 \leqq j < i \leqq m+1} \varepsilon_i \varepsilon_j b_{ij} = \sum_{1 \leqq j < i \leqq m} \varepsilon_i \varepsilon_j b_{ij} + \varepsilon_{m+1} \sum_{j=1}^{m} \varepsilon_j b_{mj}.$$

Having chosen ε_i so that the first term is $\geqq 0$, we take $\varepsilon_{m+1} = \operatorname{sgn} \Sigma_{j=1}^m \varepsilon_j b_{mj}$.

LEMMA 4. *For every $T \in B(L, L)$ there exist $(\varepsilon_u)_{u \in J_n}$, $\varepsilon_u = \pm 1$ so that*

$$|\beta_n(T) - \beta_{n-1}(T)| \leqq \sum_{A \in \Delta_n} \int_A \left| T \sum_{v \in J_n, v \subset A} \varepsilon_v 1_{\varphi_n(v)} \right| \tag{5}$$

PROOF. Obviously,

$$\beta_n(T) - \beta_{n-1}(T) = \sum_{u \in J_n} (1_u, T1_{\varphi_n(u)})$$

$$= \sum_{A \in \Delta_n} \sum_{u \in J_n, u \subset A} (1_u, T1_{\varphi_n(u)}).$$

For a fixed $A \in \Delta_n$ we apply Lemma 3 with

$$m = \operatorname{card} A = 2^n M_n^{-1}, \qquad a_{uv} = (1_u, T1_{\varphi_n(v)})$$

and we find ε_u, $u \subset A$, $u \in J_n$ so that

$$\sum_{u \subset A, u \in J_n} \left| \sum_{v \subset A, v \in J_n} \varepsilon_v \int_u T1_{\varphi_n(v)} \right| \geqq \left| \sum_{u \subset A, u \in J_n} (1_u, T1_{\varphi_n(u)}) \right|.$$

The left-hand side is equal to

$$\sum_{u \subset A, u \in J_n} \left| \int_u T \sum_{v \subset A, v \in J_n} \varepsilon_v 1_{\varphi_n(v)} \right| \leqq \sum_{u \subset A, u \in J_n} \int_u \left| T \sum_{v \subset A, v \in J_n} \varepsilon_v 1_{\varphi_n(v)} \right|$$

$$= \int_A \left| T \sum_{v \subset A, v \in J_n} \varepsilon_v 1_{\varphi_n(v)} \right|.$$

This gives the desired inequality.

Now we can easily show that the assumptions of Lemma 2 are fulfilled with

$$F_n = \bigcup_{A \in \Delta_n} \{f : f \text{ is } B_n\text{-measurable and } |f| = 1_{\varphi_n(A)}\}. \tag{6}$$

Indeed, by Lemma 4,

$$|\beta_n(T) - \beta_{n-1}(T)| \leqq M_n \max_{f \in F_n} \max_{A \in \Delta_n} \int_A |Tf| \tag{7}$$

and this gives (3).

On the other hand, if $|f| = 1_{\varphi_n(A)}$ for an $A \in \Delta_n$, then for any m, any $B \in \Delta_m$, we have, by (**),

$$\int_B |f| = \lambda(\varphi_n(A) \cap B) \leqq 4\lambda(A)\lambda(B)$$

and therefore

$$\|f\| \leqq 4\lambda(A) = 4M_n^{-1}.$$

Clearly, $\Sigma M_n^{-1} < \infty$.

It remains to define partitions Δ_n satisfying (*) and (**). We shall need the following combinatorial

LEMMA 5. *Let* card $P = 2^{2M}$, *let* $q = 2^M$. *There exist partitions* $\Omega_1, \cdots, \Omega_q$ *of* P *so that*

(*′) *all* Ω_i *consist of* q *sets of* q *elements each,*

(**′) *if* $D \in \Omega_i$, $E \in \Omega_j$, $i \neq j$, *then* $D \cap E$ *consists of one element.*

PROOF. We can treat P as an abelian field; let S be a subfield of P of order q (i.e. having q elements). Let Λ denote the set of all straight lines passing through 0, of the form xS. Evidently, Λ has $(q^2 - 1)/(q - 1) \geqq q$ elements. For each $l \in \Lambda$ let Ω_l denote the partition of P into S-lines parallel to l. Clearly, (*′) and (**′) are satisfied.

CONSTRUCTION OF Δ_n's. Now it will be more convenient to identify I with the infinite product $\nabla = \{-1, 1\}^{\aleph_0}$, equipped with the natural product measure. By this identification, B_n = the subsets of ∇ that depend on the first n coordinates only and we can put $\varphi_n(\varepsilon_1, \varepsilon_2, \cdots) = (\varepsilon_1, \varepsilon_2, \cdots, -\varepsilon_n, \cdots)$.

Let $P_n = \{-1, 1\}^{2^n}$, we represent ∇ as $\nabla = \Pi_{n=1}^{\infty} P_n$. By π_n we denote the natural projection from ∇ onto P_n. Let $q_n = \Sigma_{k=1}^{n} 2^k$ $(= 2^{n+1} - 1)$; let $\{\Omega_m\}_{m=2(q_n+1)}^{2q_{n+1}+1}$ denote a system of partitions of P_{n-1}, satisfying (*′) and (**′) of Lemma 5 (this is possible, since here $M = 2^{n-1}$, $q = 2^{2^{n-2}}$ and we need 2^{n+2} partitions), from some n on.

For $q_n < m \leqq q_{n+1}$ we set

$$\Delta_m = \{\{t \in \nabla : \pi_{n-1}(t) \in D \text{ and } t_m = 1\} : D \in \Omega_{m+1-2^n}\}$$
$$\cup \{\{t \in \nabla : \pi_{n-1}(t) \in D \text{ and } t_m = -1\} : D \in \Omega_{m+1}\}.$$

Remark that $M_m = 2^{-2^{n-2}}$ whence $m = O(2^n)$ and therefore (*) is satisfied.

Let us now check (**). So let $q_i < m \leqq q_{i+1}$, $q_j < n \leqq q_{j+1}$, let $A = \{t \in \nabla : \pi_{j+1}(t) \in D \text{ and } t_n = \varepsilon\}$, $B = \{t \in \nabla : \pi_{i-1}(t) \in E \text{ and } t_m = \eta\}$ with $D \in \Omega_{n+1-(\varepsilon+1)2^{j-1}}$, $E \in \Omega_{m+1-(\eta+1)2^{i-1}}$. Remark that $\varphi_n(A) \subset \pi_{j-1}^{-1}(D)$, $B \subset \pi_{i-1}^{-1}(E)$ and

$$\lambda(A) = \tfrac{1}{2}\lambda(\pi_{j-1}^{-1}(D)) = 2^{-2^{j-2}}$$

$$\lambda(B) = \tfrac{1}{2}\lambda(\pi_{i-1}^{-1}(E)) = 2^{-2^{i-2}}.$$

Let us consider two cases:

1. $i \neq j$. Then $\pi_{j-1}^{-1}(D)$ and $\pi_{i-1}^{-1}(E)$ depend on disjoint sets of coordinates and therefore

$$\lambda(\varphi_n(A) \cap B) \leqq \lambda(\pi_{j-1}^{-1}(D) \cap \pi_{i-1}^{-1}(E)) = \lambda(\pi_{j-1}^{-1}(D)) \cdot \lambda(\pi_{i-1}^{-1}(E))$$
$$= 4\lambda(A) \cdot \lambda(B).$$

2. $i = j$. If $m = n$ and $\varepsilon = \eta$, then $\varphi_n(A)$ and B differ on the nth ($= m$th) coordinate and are disjoint.

Otherwise D and E belong to different partitions of P_{i-1} and therefore $D \cap E$ consists of one point. Consequently,

$$\lambda(\varphi_n(A) \cap B) \leqq \lambda(\pi_{j-1}^{-1}(D \cap E)) = 2^{-2^{i-1}} = (2^{-2^{i-2}})^2 = 4\lambda(A)\lambda(B).$$

2. Applications

By a modification of the construction in the first part of this paper, we can get the following embedding theorem.

PROPOSITION 1. *Let $1 \leqq q < p \leqq \infty$. There exists a Banach lattice L which can be isometrically imbedded in $(\Sigma \oplus L_q)_{l_p}$ and which fails the approximation property.*

As an immediate corollary (by taking $1 < q < p < \infty$) we get

PROPOSITION 2. *There exists a uniformly convex Banach lattice without AP.*

Probably the most interesting applications of our result are connected with the recently introduced notion of *uniform approximation property* (*UAP*).

DEFINITION ([3]). A Banach space X is said to have λ-UAP if there exists a function $N(k)$ such that for every k-dimensional subspace E of X there exists an operator $T \in B(X,X)$ such that $T_{|E} = Id_E$, $\|T\| \leqq \lambda$ and rk $T \leqq N(k)$. *X has UAP* if there exists $\lambda < \infty$ such that X has λ-UAP.

Pełczyński and Rosenthal proved in [3] that L_p-spaces have UAP; this was extended to the case of reflexive Orlicz spaces by Lindenstrauss and Tzafriri [2].

One could therefore ask what "decent" Banach spaces have UAP. We obtain here a rather strong negative result.

PROPOSITION 3. *There exists a uniformly convex Banach space with symmetric basis, which does not have UAP.*

PROOF OF PROPOSITION 1. By an inspection of the first part of this paper we can immediately see that we obtain a lattice without AP whenever our norm $\|\ \|$ satisfies the following conditions (for some constants γ, γ_n):

$$(1') \qquad \|f\| = \||f|\|, \quad \|f\|_1 \leqq \|f\| \leqq \gamma \|f\|_\infty \quad \text{for all } f,$$

$$(4') \qquad \sum \gamma_n \max_{A \in \Delta_n} \gamma \|1_{\varphi_n(A)}\| < \infty,$$

$$(5') \qquad M_n \max_{A \in \Delta_n} \int |g| \leqq \gamma_n \|g\| \quad \text{for all } g, \text{ all } n.$$

Indeed, by setting $F'_n = \gamma_n F_n$ (the F_n from (6)) we get, by (7) and (5'),

$$(3') \qquad |\beta_n(T) - \beta_{n-1}(T)| \leqq \gamma_n \max\{\|Tf\| : f \in F_n\} = \max\{\|Tf\| : f \in F'_n\}.$$

On the other hand, (4') says exactly

$$\sum \max\{\|f\| : f \in F'_n\} < \infty.$$

We apply Lemma 2 (Lemma 1 is again valid, by (1')).

We set now tentatively

$$\|f\| = \left(\sum_{m=1}^{\infty} \sum_{B \in \Delta_m} M_m^{\alpha p} \left(\int_B |f|^q \right)^{p/q} \right)^{1/p}$$

and $\gamma_n = M_n^\beta$.

ad(4'). We have, for $A \in \Delta_n$, by (**),

$$\|1_{\varphi_n(A)}\| \leqq \left(\sum_{m=1}^{\infty} M_m \cdot M_m^{\alpha p} 4^{p/q} M_m^{-p/q} M_n^{-p/q} \right)^{1/p}$$

$$= 4^{1/q} M_n^{-1/q} \left(\sum_{m=1}^{\infty} M_m^{1+\alpha p - p/q} \right)^{1/p}$$

and (4′) is satisfied provided

$$\sum M_m^{1+\alpha p-p/q}<\infty \quad\text{and}\quad \sum M_n^{\beta-1/q}<\infty.$$

Since M_n goes very fast to ∞, it is enough if

$$1+\alpha p-\frac{p}{q}<0, \tag{8}$$

$$\beta-\frac{1}{q}<0, \tag{9}$$

ad (5′): $\|g\|\geqq M_n^{\alpha}(\int_A|g|^q)^{1/q}$ for every n and every $A\in\Delta_n$.

By Hölders inequality we get (for every n and every $A\in\Delta_n$)

$$\|g\|\geqq M_n^{\alpha}\cdot M_n^{1-1/q}\int_A|g|=M_n^{1+\alpha-1/q}\int_A|g|.$$

Again, (5′) is satisfied provided

$$\alpha+\beta-\frac{1}{q}\geqq 0. \tag{10}$$

It is easy to see that the system (8), (9), (10) is consistent if $q<p$ (it is satisfied, e.g., with $\alpha=\frac{1}{2}(1/q-1/p)$, $\beta=\frac{1}{2}(1/q+1/p)$).

It remains to prove the inequality in (1′). The left hand side inequality is satisfied provided $M_1=1$, which we can always assume. The right hand side inequality is satisfied with $\gamma=\sum M_m^{1+\alpha p-p/q}<\infty$.

This completes the proof of Proposition 1 (and of Proposition 2).

Let L be the Banach lattice from Proposition 1. Let $L^n=\{f\in L: f$ is B_n-measurable$\}$. We set $X=(\Sigma\oplus L^n)_{l_2}$; by J_n we denote the natural embedding of L^n into X. Clearly X has an unconditional basis, namely $J_n(1_{[k2^{-n},(k+1)2^{-n}]})$, $n=0,1,\cdots;\ k=0,1,\cdots,2^n-1$.

Lemma 6.[†] *X does not have UAP provided $q\geqq 2$. X is uniformly convex provided $1<q<p<\infty$.*

Proof. The second statement is obvious. To prove the first one we shall modify the proof of Lemma 1. Let $m\geqq n$, let $\lambda<\infty$ and let $T\in B(X,X)$ with $T_{|J_mL^n}=Id_{J_mL^n}$ and $\|f\|\leqq\lambda$. We want to prove that if n is big enough and m grows to ∞, then rk T must go to ∞ too.

[†] As was pointed out to the author by N. J. Nielsen and by L. Tzafriri, Lemma 6 follows trivially from the following general compactness argument:

If $X=\overline{\cup X_n}$ and X does not have UAP, then also $(\Sigma\oplus X_n)_{l_2}$ does not have UAP.

Denote by π_m the natural projection of X onto L^m. Let $T' = J_m^{-1}\pi_m T J_m : L^m \to L^m$. Clearly $\operatorname{rk} T' \leqq \operatorname{rk} T$ and $\|T'\| \leqq \|T\| \leqq \lambda$. We have also $T'|_{L^n} = Id_{L^n}$ and therefore $\beta_n(T') = 1$. Since, by (3′),

$$|\beta_m(T') - \beta_n(T')| \leqq \|T\| \sum_{j>n} \gamma_j \max\{\|1_{\varphi_j(A)}\| : A \in \Delta_j\},$$

there exists an n such that for all $m \geqq n$

$$\beta_m(T') \geqq \tfrac{1}{2}. \tag{11}$$

Let f_j be as in the proof of Lemma 1. Let $k = \operatorname{rk} T'$. There exists a number k which depends only on K, λ and γ such that there exists a $\frac{1}{4}$-net $x_1, \cdots, x_K$ in $T'(\{f \in L : \|f\| < \gamma\})$. Consequently, for every j there is an $i = 1, \cdots, K$ such that $\|Tf_j - x_i\| < \frac{1}{4}$. Therefore

$$|\beta_m(T')| \leqq K \max_{1 \leqq i \leqq K} 2^{-m} \sum_{j=1}^{2^m} |(f_j, x_i)| + \tfrac{1}{4}.$$

But since $q \geqq 2$, $\|x_i\| \geqq \|x_i\|_2$ and therefore

$$\lambda \geqq \|x_i\| \geqq \|x_i\|_2 = \left(\sum_{j=1}^{2^m} |(f_j, x_i)|^2\right)^{\frac{1}{2}} \geqq 2^{-m/2} \sum_{j=1}^{m} |(f_j, x_i)|$$

and thus $|\beta_m(T')| \leqq \frac{3}{8}$ if m is big enough. This contradicts (11).

In this way we have proved a weaker version of Proposition 3 — "symmetric" should read "unconditional". To improve this, we use a result of W. J. Davis [1]: Every uniformly convex space with unconditional basis can be embedded as a complemented subspace in a uniformly convex space with symmetric basis.

Proposition 3 follows now immediately, since UAP is hereditary with respect to complemented subspaces.

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